B.Sc. Part I (1st Paper) — Hyperbolic functions (contd.) {TRIGONOMETRY}

Q. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$

Prove that $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$

and $e^{2\phi} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

or $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$.

Soln. Given $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ —(1)

Replacing i by $-i$, we get

$\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$ —(2)

Now $\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$\Rightarrow \tan 2\theta = \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$\Rightarrow \tan 2\theta = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$\Rightarrow \tan 2\theta = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0}$$

$$\Rightarrow \tan 2\theta = \infty = \tan\frac{\pi}{2}$$

$$\Rightarrow 2\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{4}$$ (Proved)

Now, $\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$\Rightarrow \tan 2i\phi = \frac{\tan(\theta+i\phi) - \tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$\Rightarrow \tan 2i\phi = \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1+(\cos^2\alpha+\sin^2\alpha)}$$

$$\Rightarrow \tan 2i\phi = \frac{2i\sin\alpha}{2} = i\sin\alpha$$

$$\Rightarrow i\tanh 2\phi = i\sin\alpha \Rightarrow \tanh 2\phi = \sin\alpha$$

$$\Rightarrow \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo,

$$\frac{e^{2\phi} + \cancel{e^{-2\phi}} + e^{2\phi} - \cancel{e^{-2\phi}}}{\cancel{e^{2\phi}} + e^{-2\phi} - \cancel{e^{2\phi}} + e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{1-\cos(\frac{\pi}{2}+\alpha)}{1+\cos(\frac{\pi}{2}+\alpha)}$$

$$\Rightarrow \frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{2\sin^2(\frac{\pi}{4}+\frac{\alpha}{2})}{2\cos^2(\frac{\pi}{4}+\frac{\alpha}{2})} \Rightarrow e^{4\phi} = \tan^2(\frac{\pi}{4}+\frac{\alpha}{2})$$

Taking square root, we have

$$e^{2\phi} = \tan(\frac{\pi}{4}+\frac{\alpha}{2})$$

Taking log both sides, we get

$$2\phi = \log\tan(\frac{\pi}{4}+\frac{\alpha}{2})$$

$$\Rightarrow \phi = \frac{1}{2}\log\tan(\frac{\pi}{4}+\frac{\alpha}{2})$$